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Gaussian quadrature of Chebyshev polynomials

D.B. Hunter*, Geno Nikolov¹

*School of Computing and Mathematics, University of Bradford, Bradford 7, West Yorkshire,
BD7 1DP, UK*

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Abstract

We investigate the behaviour of the maximum error in applying Gaussian quadrature to the Chebyshev polynomials T_m . This quantity has applications in determining error bounds for Gaussian quadrature of analytic functions. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Suppose a weight function w is continuous and nonnegative over the interval $(-1, 1)$, and integrable over $[-1, 1]$. For a bounded integrable function f , let

$$I(f) = \int_{-1}^1 w(x)f(x) dx, \quad (1.1)$$

in particular,

$$I(1) = \int_{-1}^1 w(x) dx. \quad (1.2)$$

We can approximate $I(f)$ by a Gaussian quadrature formula

$$G_n(f) = \sum_{i=1}^n w_{n,i} f(x_{n,i}), \quad (1.3)$$

* E-mail: d.b.hunter@bradford.ac.uk.

¹ On leave from: Department of Mathematics, University of Sofia, blvd. J. Boucher 5, 1126 Sofia, Bulgaria. The author was supported by a grant from the Royal Society and by the Bulgarian Ministry of Education, Science and Technologies under Grant MM-414/94.

where $x_{n,1}, x_{n,2}, \dots, x_{n,n}$ are the zeros of the polynomial p_n of degree n in the orthogonal sequence associated with w , arranged in descending order:

$$1 > x_{n,1} > x_{n,2} > \dots > x_{n,n} > -1.$$

We shall denote the error in the approximation (1.3) by

$$E_n(f) = I(f) - G_n(f). \quad (1.4)$$

Several authors have investigated $E_n(f)$ in the case when f is a Chebyshev polynomial of the first kind, T_m , i.e.,

$$T_m(x) = \cos(m \arccos x). \quad (1.5)$$

See, e.g., [5–7]. These papers are concerned with the case $w(x) = 1$. Note that Petras also considers the Chebyshev polynomials of the second kind, U_m .

The present investigation is motivated by the following error bound, which was obtained by Hunter [4] for the case in which f is analytic in some ellipse with foci ± 1 :

$$|E_n(f)| \leq \eta_n(w) \sum_{k=0}^{\infty} |\alpha_{2n+k}|, \quad (1.6)$$

where α_i ($i = 0, 1, 2, \dots$) are the coefficients in the Chebyshev series for f , and

$$\eta_n(w) = \sup_{m \geq 2n} |E_n(T_m)|. \quad (1.7)$$

By inserting in (1.6) some bounds for the Chebyshev coefficients due to Elliott [3], we can obtain useful bounds for $|E_n(f)|$.

The determination of $\eta_n(w)$ is quite difficult, in general. However, we have the bounds

$$I(1) \leq \eta_n(w) \leq 2I(1) = \|E_n\|. \quad (1.8)$$

The norm here is the uniform norm – see [4]; the lower bound follows directly from Brass and Petras [1], Theorem 1.

In [4], Hunter showed that for the Jacobi weight function

$$w(x) = w^{(\alpha, \beta)}(x) = (1-x)^\alpha (1+x)^\beta \quad (\alpha, \beta > -1) \quad (1.9)$$

with $\alpha, \beta = \pm \frac{1}{2}$, $\eta_n(w) = I(1)$. On the basis of this result, he made the following (rather rash) conjecture.

Conjecture 1. *For all positive integers n ,*

$$\eta_n(w) = I(1).$$

Brass and Petras [1] show that this conjecture is false, in general. In fact, there is a rather obvious counterexample in the case $w(x) = 1$, when we have the result

$$\eta_1(w) = |E_1(T_4)| = \frac{32}{15}.$$

If $n > 1$, Brass and Petras found some examples, with $w(x) = 1$, for which $\eta_n(w)$ is very close to, but larger than, $I(1)$ – in fact,

$$\eta_2(w) = |E_2(T_{1515982})| = 2.00000000000003389,$$

$$\eta_3(w) = |E_3(T_{156})| = 2.0000798617910110.$$

They modified Conjecture 1, replacing it by two further conjectures.

Conjecture 2. *In the case $w(x) = 1$, $\eta_n(1) = I(1) = 2$ for all integers $n \geq 4$.*

Conjecture 3. *For the Jacobi weight function (1.9), there is a number $N_0(\alpha, \beta)$ such that*

$$\eta_n(w) = I(1) \quad \text{for all } n > N_0(\alpha, \beta).$$

In fact, Brass and Petras state Conjecture 3 for the case $\alpha = \beta$ only, but we shall obtain a partial result for the more general case below.

Our object in this paper is to investigate the behaviour of $\eta_n(w)$ for the Jacobi weight function (1.9), and for the ultraspherical weight function

$$w(x) = w^{(\lambda)}(x) = (1 - x^2)^{\lambda - 1/2} \quad (\lambda > -\frac{1}{2}). \quad (1.10)$$

It will be convenient to express the results in terms of the linear functional J given by the equation

$$J(f) = I(f)/I(1). \quad (1.11)$$

This can be approximated by a Gaussian formula

$$K_n(f) = \sum_{i=1}^n \varpi_{n,i} f(x_{n,i}), \quad (1.12)$$

where the points $x_{n,i}$ are as in (1.3), while

$$\varpi_{n,i} = w_{n,i}/I(1) \quad (i = 1, 2, \dots, n). \quad (1.13)$$

Thus,

$$\sum_{i=1}^n \varpi_{n,i} = 1. \quad (1.14)$$

The error in the approximation (1.12) is

$$\varepsilon_n(f) = J(f) - K_n(f) \quad (1.15)$$

and we shall investigate the quantity

$$\gamma_n(w) = \sup_{m \geq 2n} |\varepsilon_n(T_m)|. \quad (1.16)$$

It follows from (1.8) that

$$1 \leq \gamma_n(w) \leq 2. \quad (1.17)$$

2. The Jacobi weight function

The following theorem shows that Conjecture 3 holds in certain cases.

Theorem 2.1. *For the Jacobi weight function $w^{(\alpha,\beta)}$ given by (1.9), with $\alpha = r - 1/2$, $\beta = s - 1/2$, where r, s are nonnegative integers, we have*

$$\gamma_n(w^{(\alpha,\beta)}) = 1 \quad \text{for all integers } n > \frac{1}{2}(r + s).$$

Proof. If $m < 2n$, $E_n(T_m) = 0$, while, if $m \geq 2n > r + s$, it follows from the orthogonality properties of the Chebyshev polynomials that $J(T_m) = 0$. So, in this case, $|\varepsilon_n(T_m)| \leq 1$, and hence, from (1.17), $\gamma_n(w^{(\alpha,\beta)}) = 1$. \square

This theorem shows if α and β have the forms given, Conjecture 3 holds for the Jacobi weight function with $N_0(\alpha, \beta) = \frac{1}{2}(r + s)$. For any particular choice of r and s , it has been found that this value for $N_0(\alpha, \beta)$ can often be reduced. In fact, it has been found that Conjecture 1 holds for all pairs (r, s) of nonnegative integers with $r \leq s$, $r + s \leq 7$, apart from one exception; when $(r, s) = (1, 6)$, so that $w(x) = (1 - x)^{1/2}(1 + x)^{11/2}$, we find that

$$\gamma_1(w) = |\varepsilon_1(T_7)| = 1\,085\,341/1\,081\,344.$$

There are other pairs (r, s) for which $\gamma_1(w^{(\alpha,\beta)}) > 1$, but we have not yet found an example with $\gamma_2(w^{(\alpha,\beta)}) > 1$. \square

3. The ultraspherical weight function

For the rest of this paper, we shall be concerned with the ultraspherical weight function $w^{(\lambda)}$ given by Eq. (1.10). To emphasise this, we shall add a superscript (λ) to the quantities $I(f)$, $J(f)$, $E_n(f)$ and $\varepsilon_n(f)$, and shall write $\eta_n(w^{(\lambda)})$ and $\gamma_n(w^{(\lambda)})$ as $\eta_n^{(\lambda)}$ and $\gamma_n^{(\lambda)}$, respectively. So, e.g.,

$$I^{(\lambda)}(f) = \int_{-1}^1 (1 - x^2)^{\lambda-1/2} f(x) dx.$$

Due to symmetry, we need only consider even values $m = 2r$. The following identity is easily proved by induction:

$$I^{(\lambda)}(T_{2r}) = \frac{(-1)^r \Gamma(\lambda + \frac{1}{2}) \Gamma(\lambda + 1) \sqrt{\pi}}{\Gamma(\lambda + r + 1) \Gamma(\lambda - r + 1)}. \quad (3.1)$$

It follows immediately that

$$J^{(\lambda)}(T_{2r}) = \frac{(-1)^r [\Gamma(\lambda + 1)]^2}{\Gamma(\lambda + r + 1) \Gamma(\lambda - r + 1)} = \prod_{j=1}^r \frac{(j - 1 - \lambda)}{(j + \lambda)}. \quad (3.2)$$

Theorem 3.1. *For the ultraspherical weight function $\gamma_1^{(\lambda)} = 1$ if and only if λ is a nonnegative integer.*

Proof. It follows from (3.2) and (1.15) that

$$\varepsilon_1^{(\lambda)}(T_{2r}) = (-1)^r \left[\frac{(\Gamma(\lambda + 1))^2}{\Gamma(\lambda + r + 1)\Gamma(\lambda - r + 1)} - 1 \right].$$

If $r \leq \lambda$,

$$0 < \frac{[\Gamma(\lambda + 1)]^2}{\Gamma(\lambda + r + 1)\Gamma(\lambda - r + 1)} < 1.$$

So $|\varepsilon_1^{(\lambda)}(T_{2r})| < 1$. If $r > \lambda$, there are two cases to consider.

(i) λ is an integer. Then $J^{(\lambda)}(T_{2r}) = 0$, and hence $\varepsilon_1^{(\lambda)}(T_{2r}) = (-1)^{r+1}$, leading to the conclusion

$$\gamma_1^{(\lambda)} = 1.$$

(ii) λ is not an integer. Then if $r = [\lambda] + 2s$, (s a positive integer), $\Gamma(\lambda - r + 1)$ is negative. So

$$|\varepsilon_1^{(\lambda)}(T_{2r})| = \left| \frac{[\Gamma(\lambda + 1)]^2}{\Gamma(\lambda + r + 1)\Gamma(\lambda - r + 1)} \right| + 1 > 1 \quad (3.3)$$

and hence $\gamma_1^{(\lambda)} > 1$. \square

Corollary. For the ultraspherical weight function,

$$\gamma_1^{(\lambda)} = 1 - \frac{\lambda(\lambda - 1) \cdots (\lambda - i - 1)}{(\lambda + 1)(\lambda + 2) \cdots (\lambda + i + 2)}, \quad (3.4)$$

where $i = [\lambda]$.

Proof. If λ is an integer Eq. (3.4) gives $\gamma_1^{(\lambda)} = 1$, as required. Otherwise, if $r = i + 2s$, it follows from (3.2) and (3.3) that

$$|\varepsilon_1^{(\lambda)}(T_{2r})| = \left| \frac{\lambda(\lambda - 1) \cdots (\lambda - i - 2s + 1)}{(\lambda + 1)(\lambda + 2) \cdots (\lambda + i + 2s)} \right| + 1.$$

This has its maximum value for positive integer values of s when $s = 1$, leading to (3.4). \square

If $-\frac{1}{2} < \lambda < 0$, so that $[\lambda] = -1$, Eq. (3.4) gives the result

$$\gamma_1^{(\lambda)} = |\varepsilon_1^{(\lambda)}(T_2)| = 1/(\lambda + 1).$$

It follows that

$$\lim_{\lambda \rightarrow -1/2} \gamma_1^{(\lambda)} = 2. \quad (3.5)$$

We shall show that a similar result holds for $\gamma_n^{(\lambda)}$ with $n > 1$. Meanwhile, we propose a further conjecture.

Conjecture 4. Conjecture 1 holds for the ultraspherical weight function $w^{(\lambda)}$ if and only if λ is a nonnegative integer.

Theorem 3.2. If f is continuous over $[-1, 1]$, then, for $n \geq 2$,

$$\lim_{\lambda \rightarrow -1/2} K_n^{(\lambda)}(f) = \frac{1}{2}[f(1) + f(-1)]. \quad (3.6)$$

Proof. Since

$$I(1) = \frac{\Gamma(\lambda + \frac{1}{2})\sqrt{\pi}}{\Gamma(\lambda + 1)}$$

and

$$w_{n,i} = \frac{-2^{2-2\lambda}\pi\Gamma(n+2\lambda)}{(n+1)![\Gamma(\lambda)]^2 P_{n+1}^{(\lambda)}(x_{n,i}) P_n^{(\lambda)'}(x_{n,i})}$$

it follows that

$$\varpi_{n,i} = \frac{-2^{2-2\lambda}\sqrt{\pi}\lambda\Gamma(n+2\lambda)}{(n+1)!\Gamma(\lambda)\Gamma(\lambda + \frac{1}{2})P_{n+1}^{(\lambda)}(x_{n,i})P_n^{(\lambda)'}(x_{n,i})},$$

where we use the notation of Szego [7] for the ultraspherical polynomials $P_n^{(\lambda)}$, and $x_{n,i}$ ($i=1, 2, \dots, n$) are the zeros of $P_n^{(\lambda)}$.

It is well known that, if $n \geq 2$,

$$\lim_{\lambda \rightarrow -1/2} P_n^{(\lambda)}(x) = \frac{(1-x^2)P'_{n-1}(x)}{n(n-1)},$$

where P_{n-1} is the Legendre polynomial of degree $n-1$. It follows that

$$\lim_{\lambda \rightarrow -1/2} x_{n,1} = 1, \quad \lim_{\lambda \rightarrow -1/2} x_{n,n} = -1, \quad \lim_{\lambda \rightarrow -1/2} x_{n,i} = y_{n,i-1} \quad (i=2, 3, \dots, n-1),$$

where $y_{n,i}$ denotes the i th zero of P'_{n-1} . Hence, if $1 < i < n$,

$$\lim_{\lambda \rightarrow -1/2} \varpi_{n,i} = 0.$$

However, since

$$\lim_{\lambda \rightarrow -1/2} P_{n+1}^{(\lambda)}(x_{n,1}) = \lim_{\lambda \rightarrow -1/2} P_{n+1}^{(\lambda)}(x_{n,n}) = 0,$$

this does not apply to $\varpi_{n,1}$ and $\varpi_{n,n}$. In fact, applying Eq. (1.14) and symmetry, we get the results

$$\lim_{\lambda \rightarrow -1/2} \varpi_{n,1} = \lim_{\lambda \rightarrow -1/2} \varpi_{n,n} = \frac{1}{2}.$$

This completes the proof. \square

Corollary. For any fixed value of r , if $n \geq 2$, then

$$\lim_{\lambda \rightarrow -1/2} \varepsilon_n^{(\lambda)}(T_{2r}) = 0.$$

Proof. From Theorem 3.2 and Eq. (3.2),

$$\lim_{\lambda \rightarrow -1/2} K_n^{(\lambda)}(T_{2r}) = \lim_{\lambda \rightarrow -1/2} J^{(\lambda)}(T_{2r}) = 1.$$

The result follows immediately. \square

Despite the above corollary, there are values of λ very close to $-\frac{1}{2}$ and of r for which $\varepsilon_n^{(\lambda)}(T_{2r})$ is very close to 2.

Theorem 3.3. *For any positive integer n , there are values of $\lambda > -\frac{1}{2}$ and integers r such that $\varepsilon_n^{(\lambda)}(T_{2r})$ is arbitrarily close to 2.*

The proof depends on a number of lemmas. For $n \geq 2$, we define a quantity λ_r such that the largest zero $x_{n,1}$ of $P_n^{(\lambda_r)}$ is given by

$$x_{n,1} = \cos(\pi/2r) \text{ and, by symmetry, } x_{n,n} = -\cos(\pi/2r).$$

With this choice we have $T_{2r}(x_{n,1}) = T_{2r}(x_{n,n}) = -1$. Obviously, as r increases, $x_{n,1}$ approaches 1. By a well-known monotonicity property of the zeros of ultraspherical polynomials, we have

$$\lim_{r \rightarrow \infty} \lambda_r = -\frac{1}{2}.$$

Since the Chebyshev polynomials are uniformly bounded on $[-1, 1]$, we conclude from (3.6) that

$$\lim_{r \rightarrow \infty} K_n^{(\lambda_r)}(T_{2r}) = -1. \quad (3.7)$$

Lemma 3.1.

$$\lim_{r \rightarrow \infty} J^{(\lambda_r)}(T_{2r}) = 1.$$

To prove this, two further lemmas are required. The first is simply Eq. (6.2.16) of Szego [7].

Lemma 3.2. *Let f be a polynomial of exact degree n with real and distinct zeros. If $f(x_0) = 0$, then*

$$3(n-2)[f''(x_0)]^2 - 4(n-1)f'(x_0)f'''(x_0) \geq 0. \quad (3.8)$$

Lemma 3.3. *If $n \geq 2$, there exists a positive constant c depending on n but not on r such that*

$$\lambda_r < -\frac{1}{2} + \frac{c}{r^2}. \quad (3.9)$$

Proof. We put $f = P_n^{(\lambda_r)}$ and $x_0 = x_{n,1}$ in (3.8). Using the differential equations

$$(1-x^2)f'' - (2\lambda_r+1)xf' + n(n+2\lambda_r)f = 0,$$

$$(1-x^2)f''' - (2\lambda_r+3)xf'' + (n-1)(n+2\lambda_r+1)f' = 0$$

and the fact that $f(x_{n,1}) = 0$, after some manipulation we find that in this special case (3.8) is equivalent to the inequality

$$1 - x_{n,1}^2 \geq \frac{(2\lambda_r + 1)[(9 + 2\lambda_r)n + 4\lambda_r - 6]}{4n^3 + 4(2\lambda_r - 1)n^2 + (4\lambda_r^2 + 4\lambda_r + 5)n + 2(4\lambda_r^2 - 1)}. \quad (3.10)$$

In view of the inequalities $-\frac{1}{2} < \lambda_r < 0$ and $n \geq 2$, (3.10) yields further bounds

$$1 - x_{n,1}^2 \geq \frac{(2\lambda_r + 1)(8n - 8)}{4n^3 - 4n^2 + 5n - 2} \geq \frac{(2\lambda_r + 1)(8n - 8)}{4n^3 - 4n^2 + 8n - 8} = \frac{2(2\lambda_r + 1)}{n^2 + 2}. \quad (3.11)$$

Finally, from (3.11) and the inequality

$$\left(\frac{\pi}{2r}\right)^2 \geq \sin^2\left(\frac{\pi}{2r}\right) = 1 - x_{n,1}^2,$$

we derive the desired inequality (3.9) with

$$c = \frac{(n^2 + 2)\pi^2}{16}. \quad \square$$

Proof of Lemma 3.1 In Eq. (3.2), each factor $(j - 1 - \lambda)/(j + \lambda)$ on the right is monotonic decreasing with respect to λ for $-\frac{1}{2} < \lambda < 0$. Therefore, for sufficiently large but fixed r , we have

$$1 = J^{(-1/2)}(T_{2r}) \geq J^{(\lambda_r)}(T_{2r}) \geq J^{(-1/2+c/r^2)}(T_{2r}). \quad (3.12)$$

Furthermore, we have

$$J^{(-1/2+c/r^2)}(T_{2r}) = \prod_{j=1}^r \frac{j - 1/2 - c/r^2}{j - 1/2 + c/r^2} \geq \prod_{j=1}^r \left(1 - \frac{4c}{(2j - 1)r^2}\right) =: a_r. \quad (3.13)$$

Thus, Lemma 3.1 will be proved if we succeed in showing that

$$\lim_{r \rightarrow \infty} a_r = 1.$$

To this end, we make use of the fact that for any r numbers u_j satisfying $0 < u_j < 1$, ($j = 1, 2, \dots, r$), the following inequality holds:

$$\prod_{j=1}^r (1 - u_j) > 1 - \sum_{j=1}^r u_j.$$

For sufficiently large r , we may substitute

$$u_j = \frac{4c}{(2j - 1)r^2}$$

to obtain

$$a_r \geq 1 - \frac{4c}{r^2} \sum_{j=1}^r \frac{1}{2j - 1} =: c_r.$$

From the fact that

$$\lim_{r \rightarrow \infty} c_r = 1,$$

we infer that

$$\lim_{r \rightarrow \infty} a_r = 1,$$

completing our proof. \square

Proof of Theorem 3.3. It follows from Eq. (3.5) that the theorem holds when $n = 1$. For $n \geq 2$, it follows immediately from Lemma 3.1 and Eq. (3.7). \square

Theorem 3.3 shows that there are weight functions w for which $\gamma_n(w)$ is close to its upper bound 2. However, this behaviour appears to be untypical – in many other cases, $\gamma_n(w)$ is close to its lower bound 1. For example, in the ultraspherical case, it is easy to deduce from Eq. (3.4) that the maximum value of $\gamma_1^{(\lambda)}$ for $\lambda \geq 0$ is $4(2 - \sqrt{3}) = 1.071797$ when $\lambda = \frac{1}{2}(\sqrt{3} - 1) = 0.366025$. If $n \geq 2$, $\gamma_n^{(\lambda)}$ varies in an irregular manner with respect to λ , but is close to 1 for all values $\lambda \geq 0$. For example, the maximum value of $\gamma_2^{(\lambda)}$ for $\lambda > 0$ is 1.009484, when $\lambda = 0.2853$ – close to the value $\lambda = \frac{1}{2} \sec^2(2\pi/7) - 1$.

References

- [1] H. Brass, K. Petras, On a conjecture of D.B. Hunter, BIT 37 (1997) 227–231.
- [2] A.R. Curtis, P. Rabinowitz, On the Gaussian integration of Chebyshev polynomials, Math. Comp. 26 (1972) 207–211.
- [3] D. Elliott, The evaluation and estimation of the coefficients in the Chebyshev series expansion of a function, Math. Comp. 18 (1964) 274–284.
- [4] D.B. Hunter, Some error expansions for Gaussian quadrature, BIT 35 (1995) 64–82.
- [5] D. Nicholson, P. Rabinowitz, N. Richter, D. Zeilberger, On the error in the numerical integration of Chebyshev polynomials, Math. Comp. 25 (1971) 79–86.
- [6] K. Petras, Gaussian integration of Chebyshev polynomials and analytic functions, Numer. Algorithms 10 (1995) 187–202.
- [7] G. Szego, Orthogonal Polynomials, 4th ed., vol. 23, American Mathematical Society Colloquium Publications, Providence, RI, 1975.